Instabilities and local bifurcations. Elements of theory

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M.Haragus, G.Iooss. Local bifurcations, center manifolds, and normal forms in infinite dimensional systems (329p.). Springer UTX, 2011

Bifurcations in dimension 1

$$\frac{du}{dt} = f(u,\mu),\ \ f(0,0) = 0,\ \ \frac{\partial f}{\partial u}(0,0) = 0,\ \ \mu \text{ parameter}$$

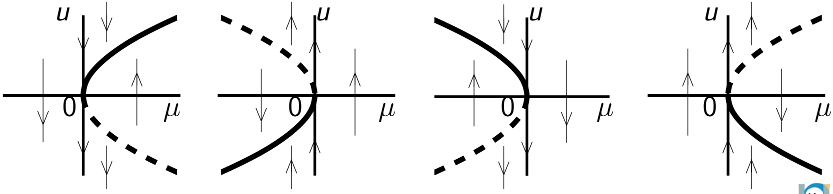
Saddle - node bifurcation

Assume f is $\mathcal{C}^k$, $k \geq 2$, in a neighborhood of $(0,0)$, and

$$\frac{\partial f}{\partial \mu}(0,0) =: a \neq 0, \quad \frac{\partial^2 f}{\partial u^2}(0,0) =: 2b \neq 0.$$

As $(u,\mu) \to (0,0)$, f has the expansion

$$f(u,\mu) = a\mu + bu^2 + o(|\mu| + u^2)$$

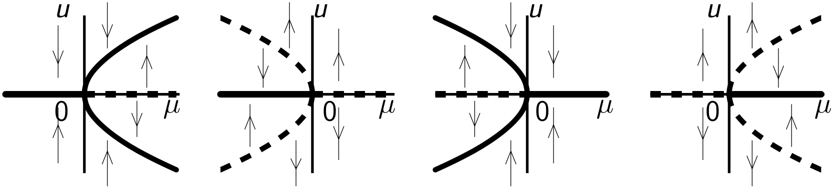


$a > 0,\ \ b < 0$ $\qquad$ $a > 0,\ \ b > 0$ $\qquad$ $a < 0,\ \ b < 0$ $\qquad$ $a < 0,\ \ b > 0$

Assume f is $\mathcal{C}^k$, $k \geq 3$, in a neighborhood of $(0,0)$, and satisfies

$$f(-u,\mu) = -f(u,\mu), \ \frac{\partial^2 f}{\partial\mu\partial u}(0,0) =: a \neq 0, \ \frac{\partial^3 f}{\partial u^3}(0,0) =: 6b \neq 0.$$

Hence, as $(u,\mu) \to (0,0)$, f has the expansion
$f(u,\mu) = a\mu u + bu^3 + o[|u|(|\mu| + u^2)]$, $u = 0$ is an equilibrium for all μ.



$a > 0, \ b < 0$ $\quad$ $a > 0, \ b > 0$ $\quad$ $a < 0, \ b < 0$ $\quad$ $a < 0, \ b > 0$

Dim 2 - Hopf bifurcation in $\mathbb{R}^2$

$$\frac{du}{dt} = \mathbf{F}(u,\mu),\ \mathbf{F}(0,0) = 0,$$

$\mathbf{F}$ is $\mathcal{C}^k$, $k \geq 3$, in a neighborhood of $(0,0)$.
Define $\mathbf{L} := D_u\mathbf{F}(0,0)$. Assume $\mathbf{L}$ has a pair of complex conjugated purely imaginary eigenvalues $\pm i\omega$, $\omega > 0$: $\mathbf{L}\zeta = i\omega\zeta,\ \mathbf{L}\overline{\zeta} = -i\omega\overline{\zeta}$.
Normal form theorem (seen later): for any integer $p \leq k$, and any μ sufficiently small, there exists a polynomial $\mathbf{\Phi}_\mu$ of degree p in $(A, \overline{A})$, with complex coefficients functions of μ, taking values in $\mathbb{R}^2$, such that

$$\mathbf{\Phi}_0(0,0) = 0, \quad \partial_A\mathbf{\Phi}_0(0,0) = 0, \quad \partial_{\overline{A}}\mathbf{\Phi}_0(0,0) = 0,$$

$$u = A\zeta + \overline{A\zeta} + \mathbf{\Phi}_\mu(A,\overline{A}),\ A \in \mathbb{C},$$

transforms the system into the differential equation

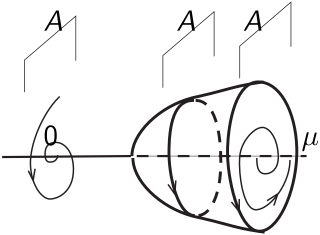
$$\frac{dA}{dt} = i\omega A + AQ(|A|^2,\mu) + o(|A|^p),\ Q \text{ polynomial in } |A|^2, Q(0,0) = 0$$

$$\frac{dA}{dt} = i\omega A + A(a\mu + b|A|^2) + o(|A|(|\mu| + |A|^2)),$$

Assume $a_r \neq 0$ and $b_r \neq 0$.

Truncated system: set $A = re^{i\phi}$,

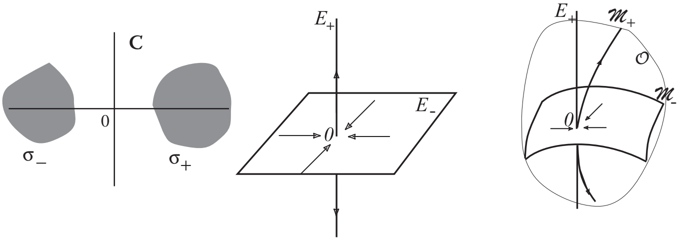
$$\begin{aligned} \frac{dr}{dt} &= r(a_r\mu + b_r r^2) \text{ (pitchfork bifurcation for radial part)} \\ \frac{d\phi}{dt} &= \omega + a_i\mu + b_i r^2, \text{ (frequency of bifurcated periodic solution)} \end{aligned}$$



case $a_r > 0$, $b_r < 0$

Hyperbolic situation in $\mathbb{R}^n$

$$\frac{du}{dt} = F(u), \; F(0) = 0, \; DF(0) = L$$



left: spectrum of L, center: linear situation, right: nonlinear situation

Hyperbolic situation in $\mathbb{R}^n$ continued

$$u = X + Y,\ X = P_+ u \in E_+, Y = P_- u \in E_-$$

$$\begin{aligned} \frac{dX}{dt} &= L_+ X + P_+ R(X+Y) \\ \frac{dY}{dt} &= L_- Y + P_- R(X+Y) \end{aligned}$$

Unstable manifold $\mathcal{M}_+$: solve in $u(t), t \leq 0$, with $u(t) \to 0$ as $t \to -\infty$

$$u(t) = e^{L_+ t} X + \int_0^t e^{L_+(t-s)} P_+ R(u(s)) ds + \int_{-\infty}^t e^{L_-(t-s)} P_- R(u(s)) ds$$

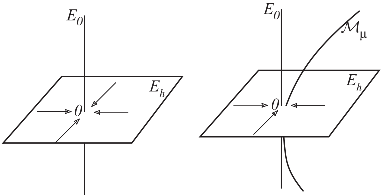
Then, by implicit function theorem, $u(t) = \Phi_+(X, t)$,
and $u(0) = \Phi_+(X, 0) = X + \Psi_+(X)$, with $\Psi_+(X) \in E_-$

Center manifold in $\mathbb{R}^n$

Pliss 1964, Kelley 1967, Lanford 1973, Henry 1981, Mielke 1988, Kirrmann 1991, Vanderbauwhede - Iooss 1992

$$\frac{du}{dt} = Lu + R(u,\mu),\ (u,\mu) \in \mathbb{R}^n \times \mathbb{R}^m, R(0,0) = 0,\ D_u R(0,0) = 0.$$

spectrum of $L = \sigma = \sigma_- \cup \sigma_0$
Hypothesis: $\sigma_0 =$ finite number of eigenvalues of finite mutiplicities
$sup_{\lambda\in\sigma_-}\lambda < -\gamma < 0$ (gap assumption)
$\mathbb{R}^n = E_0 \oplus E_-,\ u = X + Y,\ X = P_0 u, Y = P_- u$



left: linear case for $\mu = 0$, asymptotic solutions $\in E_0$, right: non linear case

Center Manifolds- idea of proof

Theorem:

$$\begin{aligned} \mathcal{M}_\mu &= \{u = u_0 + \mathbf{\Psi}(u_0, \mu),\ (u_0, \mu) \in E_0 \times \mathbb{R}^m\} \\ \mathbf{\Psi} &\in \mathcal{C}^k(\mathcal{O}_0, E_-),\ \mathcal{O}_0 \text{ neighb of 0 in } E_0 \times \mathbb{R}^m \\ & \mathbf{\Psi}(0,0) = 0, D_{u_0}\mathbf{\Psi}(0,0) = 0. \end{aligned}$$

$\mathcal{M}_\mu$ locally invariant and *locally attracting*.

Idea of proof: Even though $u(t)$ stays bounded for $t \in \mathbb{R}$, the first term and the integral below with L_0 may grow polynomially in t as $t \to -\infty$.

$$u(t) = e^{L_0 t}X + \int_0^t e^{L_0(t-s)}P_0 R(u(s))ds + \int_{-\infty}^t e^{L_-(t-s)}P_- R(u(s))ds.$$

Need of a (smooth) "cut-off" function on E_0, modifying and making the system linear for its part in E_0, outside a ball of small radius. This allows to work in a space of functions growing at infinity.

New complications due to the fact that we deal with such functions (which may grow at $-\infty$ with a small exponential).

Center Manifolds in infinite dimensions

$$\frac{du}{dt} = Lu + R(u, \mu)$$

$$R(0,0) = 0, \; D_u R(0,0) = 0$$

L linear bounded $\mathcal{Z} \to \mathcal{X}$,
$\mathcal{Z}$ cont. embedded in $\mathcal{X}$ (both Hilbert spaces)
$R : (\mathcal{Z} \times \mathbb{R}^m) \to \mathcal{X}$ of class $\mathcal{C}^k, k \geq 2$ in a neighborhood of 0
Hypothesis:
(i) (gap assumption) spectrum σ of $L = \sigma_0 \cup \sigma_-$,
For $\lambda \in \sigma_0, Re\lambda = 0$,
$sup_{\lambda \in \sigma_-} Re\lambda < -\gamma < 0$;
(ii) $\sigma_0 =$ finite number of eigenvalues of finite mutiplicities

Center Manifolds in infinite dimensions - continued

Hypothesis on the linearized system

$$||(i\omega\mathbb{I} - L)^{-1}||_{\mathcal{L}(\mathcal{X})} \leq \frac{C}{|\omega|} \text{ for } \omega \in \mathbb{R}, |\omega| \text{ large.}$$

Then the following properties (iii) and (iv) are satisfied.
Define:

$\mathcal{E}_0 = P_0\mathcal{X} = P_0\mathcal{Z},\ \mathcal{Z}_h = P_h\mathcal{Z}, \mathcal{X} = \mathcal{E}_0 \oplus \mathcal{X}_h, \mathcal{Z} = \mathcal{E}_0 \oplus \mathcal{Z}_h, \eta \in [0, \gamma]$

(iii) $\frac{du_h}{dt} = L_h u_h + f,\ f \in C^0(\mathbb{R}, \mathcal{X}), sup_{t\in\mathbb{R}} e^{\eta t}||f(t)||_{\mathcal{X}} < \infty,$
Then, there exists a unique $u_h = K_h f$, such that
$K_h f \in C^0(\mathbb{R}, \mathcal{Z}), sup_{t\in\mathbb{R}} e^{\eta t}||K_h f(t)||_{\mathcal{Z}} < C(\eta) sup_{t\in\mathbb{R}} e^{\eta t}||f(t)||_{\mathcal{X}},$
$C(\eta)$ continuous on $[0, \gamma]$.

(iv) $\frac{du_h}{dt} = L_h u_h, u|_{t=0} \in \mathcal{Z}_h.$
Then, there exists a unique $u_h \in C^0(\mathbb{R}^+, \mathcal{Z}_h), ||u_h||_{\mathcal{Z}} \leq c_\eta e^{-\eta t}, t \geq 0.$

Reduced system for asymptotic dynamics and Symmetries

$$\begin{aligned}\frac{du_0}{dt} &= L_0 u_0 + P_0 R(u_0 + \boldsymbol{\Psi}(u_0,\mu),\mu) := f(u_0,\mu) \\ f(0,0) &= 0, D_{u_0} f(0,0) = L_0, \text{ spectrum of } L_0 : \sigma_0\end{aligned}$$

Frequent case: 0 is a solution of the system for any μ
$R(0,\mu) = 0$, hence $\boldsymbol{\Psi}(0,\mu) = 0, f(0,\mu) = 0$ and

the linear operator $A_\mu := D_{u_0} f(0,\mu)$ has the eigenvalues close to the imaginary axis of the linearized operator $L_\mu := L + D_u R(0,\mu)$

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Presence of symmetry

$$\begin{aligned} TLu &= LTu,\ TR(u,\mu) = R(Tu,\mu)\\ T|_{\mathcal{E}_0} &:= T_0 \quad \text{is an isometry}\end{aligned}$$

Then

$$\begin{aligned} T\mathbf{\Psi}(u_0,\mu) &= \mathbf{\Psi}(T_0 u_0,\mu),\ \text{for } u_0 \in \mathcal{E}_0\\ T_0 f(u_0,\mu) &= f(T_0 u_0,\mu).\end{aligned}$$

Computation of center manifold and reduced system

NB. We compute Taylor expansions, in powers of $(u_0, \mu) \in \mathcal{E}_0 \times \mathbb{R}^m$

$$D_{u_0}\mathbf{\Psi}(u_0,\mu)\frac{du_0}{dt} = \frac{du_h}{dt}$$

replace $\frac{du_0}{dt}$ by $L_0 u_0 + P_0 R(u_0 + \mathbf{\Psi}(u_0,\mu),\mu)$,
and replace $\frac{du_h}{dt}$ by $L_h \mathbf{\Psi}(u_0,\mu) + P_h R(u_0 + \mathbf{\Psi}(u_0,\mu),\mu)$
and *identify powers of* (u_0, μ).

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and *identify powers of* (u_0,μ).
Example: quadratic order in u_0:
$f(u_0,\mu) = L_0 u_0 + P_0 R_{2,0}(u_0) + P_0 R_{0,1}(\mu) + h.o.t.$, h.o.t. depends on $\mathbf{\Psi}$
$D_{u_0}\mathbf{\Psi}_{2,0}(u_0) L_0 u_0 - L_h \mathbf{\Psi}_{2,0}(u_0) = P_h R_{2,0}(u_0)$ leads to
$\mathbf{\Psi}_{2,0}(u_0) = \int_0^\infty e^{L_h t} P_h R_{2,0}(e^{-L_0 t} u_0) dt$.
This may become tedious, and may lead to a complicate vector field in $\mathcal{E}_0$, in case of dimension > 1, specially if orders > 2 are required.
Our purpose now is to simplify the reduced system, in using Symmetries and Normal form theory.

Normal forms

Poincaré, Birkhoff, Arnold, Belitskii, Elphick et al...
$p \geq 2$, $\exists$ polynomial $\Phi_\mu : \mathcal{E}_0 \to \mathcal{E}_0$, of degree p and a neighborhood $\mathcal{O}_0$ of 0 in $\mathcal{E}_0 \times \mathbb{R}^m$, such that the local change of variable in $\mathcal{E}_0$

$$u_0 = v_0 + \Phi_\mu(v_0)$$

transforms the reduced system into a new system where $\mathbf{N}_\mu$ is a polynomial of degree p such that

$$\frac{dv_0}{dt} = L_0 v_0 + \mathbf{N}_\mu(v_0) + \rho(v_0, \mu),$$

$$\begin{aligned} \mathbf{N}_0(0) &= 0, \quad D_{v_0}\mathbf{N}_0(0) = 0 \\ e^{L_0^* t}\mathbf{N}_\mu(v_0) &= \mathbf{N}_\mu(e^{L_0^* t} v_0), \forall (t, v_0) \in \mathbb{R} \times \mathcal{E}_0, \\ \rho(v_0, \mu) &= o(||v_0||^p). \end{aligned}$$

NB. In case of analytical vector fields, there are results optimizing the degree p, giving a rest ρ exponentially small (G.I., E.Lombardi 2005)

Normal forms - continued

Equivalent characterization:

$$D_{v_0}\mathbf{N}_\mu(v)L_0^* v = L_0^*\mathbf{N}_\mu(v) \text{ for all } v \in \mathcal{E}_0 \text{ and } \mu \in \mathbb{R}^m$$

Normal forms - continued

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Case of a linear operator $L + R_\mu,\ R_0 = 0$
interesting when L is not diagonalizable.
Then Φ_μ is linear (only degree 1 terms); the normal form $L + \mathbf{N}_\mu$ is also linear, and

$$\mathbf{N}_\mu L^* = L^*\mathbf{N}_\mu.$$

Normal forms - continued

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Cases with Symmetries Assume that the nonlinear system is *equivariant under an isometry* T in $\mathbb{R}^n$
Then, polynomials $\mathbf{N}_\mu$ and Φ_μ commute with T.

Normal forms - idea of proof

$du/dt = Lu + R(u)$, $u \in \mathbb{R}^n$, p is a given number ≥ 2. No μ here, for simplification.

$$R(u) = \sum_{2 \leq q \leq p} R_q(u^{(q)}) + o(||u||^p),$$

R_q is q-linear symmetric on $(\mathbb{R}^n)^q$. Analogous notation for Φ_q and $\mathbf{N}_q$. Differentiate $u = v + \Phi(v)$ with respect to t, and replace du/dt and dv/dt:

$$(\mathbb{I} + D\Phi(v))(Lv + \mathbf{N}(v) + \rho(v)) = L(v + \Phi(v)) + R(v + \Phi(v))$$

Identify powers of v:

$$\begin{aligned} \mathcal{A}_L \Phi_q &= Q_q - \mathbf{N}_q, q = 2, 3, ...p; \;\; Q_2 = R_2 \\ \mathcal{A}_L \Phi(v) : &= D\Phi(v)Lv - L\Phi(v) \text{ for all } v \in \mathbb{R}^n. \end{aligned}$$

$Q_q - \mathbf{N}_q \in ker(\mathcal{A}_{L^*})^\perp$, i.e. we can choose $\mathbf{N}_q = P_{ker(\mathcal{A}_{L^*})} Q_q$, and $\Phi_q \in ker(\mathcal{A}_L)^\perp$ (makes the solution uniquely determined).

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Computation of Center Manifold and Normal form

Center manifold theorem gives

$$u = u_0 + \mathbf{\Psi}(u_0, \mu), u_0 \in \mathcal{E}_0 \text{ and } \mathbf{\Psi}(u_0, \mu) \in \mathcal{Z}_h$$

Normal form applied to the reduced system for $u_0 \in \mathcal{E}_0$:

$$u_0 = v_0 + \mathbf{\Phi}_\mu(v_0), \ \frac{dv_0}{dt} = L_0 v_0 + \mathbf{N}_\mu(v_0) + \boldsymbol{\rho}(v_0, \mu).$$

Consequently, we can write

$$u = v_0 + \widetilde{\mathbf{\Psi}}(v_0, \mu),$$

with

$$\widetilde{\mathbf{\Psi}}(v_0, \mu) = \mathbf{\Phi}_\mu(v_0) + \mathbf{\Psi}(v_0 + \mathbf{\Phi}_\mu(v_0), \mu) \in \mathcal{Z}.$$

$\widetilde{\mathbf{\Psi}}(v_0, \mu)$ belongs to the entire space $\mathcal{Z}$, and not to $\mathcal{Z}_h$.

Computation of Center Manifold and Normal form - continued

Differentiating with respect to t and replacing du/dt and dv_0/dt, leads to

$$D_{v_0}\widetilde{\mathbf{\Psi}}(v_0,\mu)L_0 v_0 - L\widetilde{\mathbf{\Psi}}(v_0,\mu) + \mathbf{N}_\mu(v_0) = Q(v_0,\mu),$$

$$Q(v_0,\mu) = \mathbf{\Pi}_p\left(R(v_0+\widetilde{\mathbf{\Psi}}(v_0,\mu),\mu) - D_{v_0}\widetilde{\mathbf{\Psi}}(v_0,\mu)\mathbf{N}_\mu(v_0)\right).$$

$\mathbf{\Pi}_p$ represents the linear map that associates to a map of class $\mathcal{C}^p$ the polynomial of degree p in its Taylor expansion.

Projecting on $\mathcal{E}_0$ and $\mathcal{Z}_h$ gives :

$$\begin{aligned} \mathcal{A}_{L_0}\widetilde{\mathbf{\Psi}}_0(v_0,\mu) + \mathbf{N}_\mu(v_0) &= Q_0(v_0,\mu) \\ D_{v_0}\widetilde{\mathbf{\Psi}}_h(v_0,\mu)L_0 v_0 - \mathbf{L}_h\widetilde{\mathbf{\Psi}}_h(v_0,\mu) &= Q_h(v_0,\mu), \end{aligned}$$

where

$$Q_0(v_0,\mu) = P_0 Q(v_0,\mu), \quad Q_h(v_0,\mu) = P_h Q.$$

Example: Hopf bifurcation

$\sigma_0 = \{\pm i\omega\},\ L_0\zeta = i\omega\zeta,\ \mu \in \mathbb{R}$

$$u = v_0 + \mathbf{\Psi}_\mu(v_0),\ \mathbf{\Psi}_\mu(v_0) \in \mathcal{Z}$$

For $v_0(t) \in \mathcal{E}_0$, it is convenient to write

$$v_0(t) = A(t)\zeta + \overline{A(t)\zeta}, \quad A(t) \in \mathbb{C},$$

and since $\mathbf{N}_\mu(A, \overline{A}) = (AQ(|A|^2, \mu), \overline{AQ}(|A|^2, \mu))$, the reduced system reads

$$\frac{dA}{dt} = i\omega A + AQ(|A|^2, \mu) + \rho(A, \overline{A}, \mu)$$

Q complex-valued, polynomial in its first argument, with $Q(0,0) = 0$.
We need to compute coefficients a and b in

$$Q(|A|^2, \mu) = a\mu + b|A|^2 + O((|\mu| + |A|^2)^2).$$

$$\mathbf{\Psi}_{ql}(v_0^{(q)}, \mu^{(l)}) = \mu^l \sum_{q_1+q_2=q} A^{q_1}\overline{A}^{q_2}\mathbf{\Psi}_{q_1q_2l}, \quad \mathbf{\Psi}_{q_1q_2l} \in \mathcal{Z}.$$

By identifying the terms of order $O(\mu)$, $O(A^2)$, and $O(A\overline{A})$, we obtain

$$\begin{aligned} -L\mathbf{\Psi}_{001} &= R_{01}, \\ (2i\omega - L)\mathbf{\Psi}_{200} &= R_{20}(\zeta,\zeta), \\ -L\mathbf{\Psi}_{110} &= 2R_{20}(\zeta,\overline{\zeta}). \end{aligned}$$

Operators L and $(2i\omega - L)$ are invertible, so that $\mathbf{\Psi}_{001}$, $\mathbf{\Psi}_{200}$, and $\mathbf{\Psi}_{110}$ are uniquely determined . Next, identify the terms of order $O(\mu A)$ and $O(A^2\overline{A})$:

$$\begin{aligned} (i\omega - L)\mathbf{\Psi}_{101} &= -a\zeta + R_{11}(\zeta) + 2R_{20}(\zeta, \mathbf{\Psi}_{001}), \\ (i\omega - L)\mathbf{\Psi}_{210} &= -b\zeta + 2R_{20}(\zeta, \mathbf{\Psi}_{110}) + 2R_{20}(\overline{\zeta}, \mathbf{\Psi}_{200}) + 3R_{30}(\zeta,\zeta,\overline{\zeta}). \end{aligned}$$

Example: Hopf bifurcation - continued 2

The range of $(i\omega - L)$ is of codimension 1, so we can solve these equations and determine $\mathbf{\Psi}_{101}$ and $\mathbf{\Psi}_{200}$, provided the right hand sides satisfy one solvability condition.
The solvability condition is that the right hand sides be orthogonal to the kernel of the adjoint $(-i\omega - L^*)$ of $(i\omega - L)$. The kernel of $(-i\omega - L^*)$ is spanned by $\zeta^* \in \mathcal{X}^*$ that we choose such that $\langle \zeta, \zeta^* \rangle = 1$. Then

$$\begin{aligned} a &= \langle R_{11}(\zeta) + 2R_{20}(\zeta, \mathbf{\Psi}_{001}), \zeta^* \rangle, \\ b &= \langle 2R_{20}(\zeta, \mathbf{\Psi}_{110}) + 2R_{20}(\overline{\zeta}, \mathbf{\Psi}_{200}) + 3R_{30}(\zeta, \zeta, \overline{\zeta}), \zeta^* \rangle. \end{aligned}$$

Example: Hopf bifurcation with $O(2)$ symmetry

Assume that we have a group $\{\mathbf{R}_\varphi, \mathbf{S}; \varphi \in \mathbb{R}/2\pi\mathbb{Z}\}$ representation of an $O(2)$ symmetry in $\mathcal{X}$ and $\mathcal{Z}$: we have $\mathbf{S}$, and $\mathbf{R}_\varphi$ with $\mathbf{S}^2 = \mathbb{I}$, and

$$\begin{aligned}
\mathbf{R}_\varphi \mathbf{S} &= \mathbf{S}\mathbf{R}_{-\varphi} \text{ for all } \varphi \in \mathbb{R}/2\pi\mathbb{Z} \\
\mathbf{R}_\varphi \circ \mathbf{R}_\psi &= \mathbf{R}_{\varphi+\psi} \text{ for all } \varphi,\ \psi \in \mathbb{R}/2\pi\mathbb{Z} \\
\mathbf{R}_0 &= \mathbb{I}
\end{aligned}$$

Assume that our system commutes with this representation of $O(2)$:

$$\mathbf{S}L = L\mathbf{S},\ \ R(\mathbf{S}u, \mu) = \mathbf{S}R(u, \mu) \text{ for all } \mu \in \mathbb{R}$$

and $\mathbf{R}_\varphi L = L\mathbf{R}_\varphi$,$R(\mathbf{R}_\varphi u, \mu) = \mathbf{R}_\varphi R(u, \mu)$ for all $\varphi \in \mathbb{R}/2\pi\mathbb{Z}$, $u \in \mathcal{Z}$, and $\mu \in \mathbb{R}$.

Hopf bifurcation with $O(2)$ symmetry - continued 1

Assume that $\sigma_0 = \{\pm i\omega\}$, and eigenvectors are not invariant under the action of $\mathbf{R}_\varphi$.

Notice that any eigenvalue λ of L that has an eigenvector ζ not invariant under the action of $\mathbf{R}_\varphi$ is at least geometrically double.

Generically, $\pm i\omega$ are algebraically and geometrically double eigenvalues.

Then the restriction of the action of $\mathbf{R}_\varphi$ to the eigenspaces associated with the eigenvalues $\pm i\omega$ is not trivial, and we can choose the eigenvectors $\{\zeta_0, \zeta_1\}$ associated with $i\omega$ such that

$$\mathbf{R}_\varphi \zeta_0 = e^{im\varphi}\zeta_0, \quad \mathbf{R}_\varphi \zeta_1 = e^{-im\varphi}\zeta_1, \quad \mathbf{S}\zeta_0 = \zeta_1, \quad \mathbf{S}\zeta_1 = \zeta_0.$$

$\{\overline{\zeta}_0, \overline{\zeta}_1\}$ are the eigenvectors associated with $-i\omega$.

Hopf bifurcation with $O(2)$ symmetry - Normal form

$$
\begin{aligned}
u &= v_0 + \widetilde{\mathbf{\Psi}}(v_0,\mu), \quad v_0 \in \mathcal{E}_0, \quad \widetilde{\mathbf{\Psi}}(v_0,\mu) \in \mathcal{Z}, \\
v_0(t) &= A(t)\zeta_0 + B(t)\zeta_1 + \overline{A(t)\zeta_0} + \overline{B(t)\zeta_1}.
\end{aligned}
$$

$\widetilde{\mathbf{\Psi}}(\cdot,\mu)$ commutes with $\mathbf{R}_\varphi$ and $\mathbf{S}$. Define $\mathbf{N}_\mu = (\Phi_0, \Phi_1, \overline{\Phi}_0, \overline{\Phi}_1)$, where Φ_j, $j = 0, 1$, are polynomials of $(A, B, \overline{A}, \overline{B})$ with coefficients depending upon μ. Using successively the characterization theorem and the fact that $\mathbf{N}_\mu$ commutes with $\mathbf{R}_\varphi$ and $\mathbf{S}$, we find that

$$
\begin{aligned}
\Phi_0(e^{-i\omega t}A, e^{-i\omega t}B, e^{i\omega t}\overline{A}, e^{i\omega t}\overline{B}) &= e^{-i\omega t}\Phi_0(A, B, \overline{A}, \overline{B}), \\
\Phi_1(e^{-i\omega t}A, e^{-i\omega t}B, e^{i\omega t}\overline{A}, e^{i\omega t}\overline{B}) &= e^{-i\omega t}\Phi_1(A, B, \overline{A}, \overline{B}), \\
\Phi_0(e^{im\varphi}A, e^{-im\varphi}B, e^{-im\varphi}\overline{A}, e^{im\varphi}\overline{B}) &= e^{im\varphi}\Phi_0(A, B, \overline{A}, \overline{B}), \\
\Phi_1(e^{im\varphi}A, e^{-im\varphi}B, e^{-im\varphi}\overline{A}, e^{im\varphi}\overline{B}) &= e^{-im\varphi}\Phi_1(A, B, \overline{A}, \overline{B}), \\
\Phi_0(B, A, \overline{B}, \overline{A}) &= \Phi_1(A, B, \overline{A}, \overline{B})
\end{aligned}
$$

for all $t \in \mathbb{R}$ and $\varphi \in \mathbb{R}/2\pi\mathbb{Z}$.

Hopf bifurcation with $O(2)$ symmetry - Normal form-continued

$$\begin{aligned}\frac{dA}{dt} &= i\omega A + A(a\mu + b|A|^2 + c|B|^2) + \rho(A, B, \overline{A}, \overline{B}, \mu)\\ \frac{dB}{dt} &= i\omega B + B(a\mu + b|B|^2 + c|A|^2) + \rho(B, A, \overline{B}, \overline{A}, \mu),\end{aligned}$$

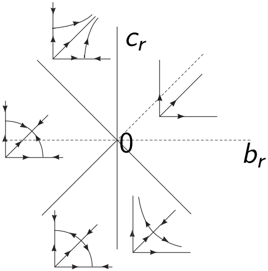
with $\rho(A, B, \overline{A}, \overline{B}, \mu) = O((|A| + |B|)(|A|^2 + |B|^2 + |\mu|)^2)$.

$A = r_0 e^{i\theta_0}, \quad B = r_1 e^{i\theta_1},$ then for the truncated system

$$\begin{aligned}\frac{d\theta_0}{dt} &= \omega + a_i\mu + b_i r_0^2 + c_i r_1^2,\\ \frac{d\theta_1}{dt} &= \omega + a_i\mu + b_i r_1^2 + c_i r_0^2.\end{aligned}$$

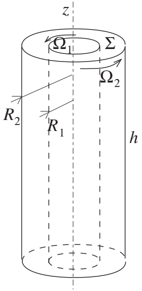
Hopf bifurcation with $O(2)$ symmetry - Dynamics

$$\frac{dr_0}{dt} = r_0(a_r\mu + b_r r_0^2 + c_r r_1^2),$$
$$\frac{dr_1}{dt} = r_1(a_r\mu + b_r r_1^2 + c_r r_0^2),$$



phase portraits in the (r_0, r_1) plane, in the case $a_r\mu > 0$. For $b_r < 0$ two pairs of equilibria $(\pm r_*(\mu), 0)$ and $(0, \pm r_*(\mu))$ corresponding to *rotating waves*. For $b_r + c_r < 0$ pair of equilibria with $r_0 = r_1$, corresponding to *standing waves*.

Couette - Taylor hydrodynamic problem



$$\frac{\partial V}{\partial t} + (V \cdot \nabla)V + \frac{1}{\rho}\nabla p = \nu \Delta V, \ \nabla \cdot V = 0, \ + \text{Boundary Cond.}$$

Couette flow In cylindrical coordinates (r, θ, z)

$$V^{(0)} = (0, v_0(r), 0), \quad p^{(0)} = \rho \int \frac{v_0^2}{r} \, dr$$

$$v_0(r) = \frac{\Omega_2 R_2^2 - \Omega_1 R_1^2}{R_2^2 - R_1^2} r + \frac{(\Omega_1 - \Omega_2) R_1^2 R_2^2}{R_2^2 - R_1^2} \frac{1}{r}.$$

Couette - Taylor problem (2)

We set $V = V^{(0)} + U, \ p = p^{(0)} + \rho q,$

$$\begin{aligned} \frac{\partial U}{\partial t} &= \nu \Delta U - (V^{(0)} \cdot \nabla)U - (U \cdot \nabla)V^{(0)} - (U \cdot \nabla)U - \nabla q \\ \nabla \cdot U &= 0, U|_{r=R_1,R_2} = 0 \end{aligned}$$

Periodicity condition in the axis direction:
$U(x + he_z, t) = U(x, t), \nabla p(x, t) = \nabla p(x + he_z, t)$ completed by a zero flux condition through any section of the cylindrical domain.

$$\mathcal{X} = \left\{ U \in \left(L^2(\Sigma \times (\mathbb{R}/h\mathbb{Z}))\right)^3; \nabla \cdot U = 0, \ U \cdot n|_{\partial\Sigma\times\mathbb{R}} = 0, \ \int_\Sigma U \cdot n \, dS = 0$$

$$\mathcal{Z} = \left\{ U \in \mathcal{X}; U \in \left(H^2(\Sigma \times (\mathbb{R}/h\mathbb{Z}))\right)^3, U|_{\partial\Sigma\times\mathbb{R}} = 0 \right\}$$

The orthogonal complement of $\mathcal{X}$ in $\left(L^2(\Sigma \times (\mathbb{R}/h\mathbb{Z}))\right)^3$ is the space $\{\nabla\phi; \phi \in H^1(\Sigma \times (\mathbb{R}/h\mathbb{Z})) + z\mathbb{R}\}$, i.e., $\nabla\phi$ is a periodic function, while ϕ is not periodic .

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Couette - Taylor problem (3)

$$\frac{dU}{dt} = \mathbf{L}U + \mathbf{R}(U), \text{ in } \mathcal{X} \text{ for } U(\cdot, t) \in \mathcal{Z}$$

$$\mathbf{L}U = \mathbf{\Pi}_0 \left(\nu \Delta U - (V^{(0)} \cdot \nabla)U - (U \cdot \nabla)V^{(0)}\right), \quad \mathbf{R}(U) = -\mathbf{\Pi}_0 \left((U \cdot \nabla)U\right).$$

Representations of symmetries commuting with the system

$$\begin{aligned} (\boldsymbol{\tau}_a U)(r,\theta,z) &= U(r,\theta,z+a), \ a \in \mathbb{R}/h\mathbb{Z}, \\ (\mathbf{S}U)(r,\theta,z) &= (U_r(r,\theta,-z), U_\theta(r,\theta,-z), -U_z((r,\theta,-z)), \\ (\mathbf{R}_\phi U)(r,\theta,z) &= U(r,\theta+\phi,z), \ \phi \in \mathbb{R}/2\pi\mathbb{Z}, \end{aligned}$$

satisfy ($O(2)$ action)

$$\boldsymbol{\tau}_a \mathbf{S} = \mathbf{S}\boldsymbol{\tau}_{-a}, \quad \boldsymbol{\tau}_h = \mathbb{I}, \quad \boldsymbol{\tau}_a \boldsymbol{\tau}_b = \boldsymbol{\tau}_{a+b}.$$

$\mathbf{R}_\phi$ represents a $SO(2)$ action, which commutes with the $O(2)$ action.

Couette - Taylor problem (4)

Three dimensionless parameters appear in the equations:

$$\Omega_r = \frac{\Omega_2}{\Omega_1}, \quad \eta = \frac{R_1}{R_2}, \quad \mathcal{R} = \frac{R_1\Omega_1(R_2 - R_1)}{\nu}$$

Fixing Ω_r and η, we take $\mathcal{R}$ as bifurcation parameter, and denote $\mathbf{L}$ by $\mathbf{L}_{\mathcal{R}}$. For low values of $\mathcal{R}$, the spectrum of $\mathbf{L}_{\mathcal{R}}$ is strictly contained in the left half-complex plane, i.e., the Couette flow is stable.

Instabilities are obtained by increasing $\mathcal{R}$ (for instance by increasing the rotation rate of the inner cylinder).

The Case $\Omega_r > 0$ or $\Omega_r < 0$ Close to 0

In this case it has been shown numerically that as $\mathcal{R}$ increases, there is a critical value $\mathcal{R}_c$ for which an eigenvalue of $\mathbf{L}_{\mathcal{R}}$ crosses the imaginary axis, passing through 0 from the left to the right, and all other eigenvalues remain in the left half-complex plane.

0 is a double eigenvalue with complex conjugated eigenvectors

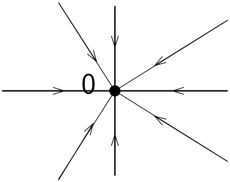
$$\zeta = e^{ik_c z}\widehat{U}(r), \; \overline{\zeta} = \mathbf{S}\zeta, \; \boldsymbol{\tau}_a\zeta = e^{ik_c a}\zeta \text{ for all } a \in \mathbb{R}.$$

Couette - Taylor problem (5)

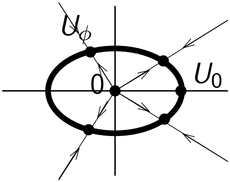
Two-dimensional center manifold: $U = A\zeta + \overline{A\zeta} + \mathbf{\Psi}(A, \overline{A}, \mu)$
Reduced system in $\mathbb{C}$: $\frac{dA}{dt} = f(A, \overline{A}, \mu)$

$$\text{Symmetries:} \quad \begin{aligned} f(\overline{A}, A, \mu) &= \overline{f(A, \overline{A}, \mu)} \\ f(e^{ik_c a}A, e^{-ik_c a}\overline{A}, \mu) &= e^{ik_c a} f(A, \overline{A}, \mu), \text{ for any } a \in \mathbb{R} \end{aligned}$$

Then $\frac{dA}{dt} = Ag(|A|^2, \mu) = \alpha\mu A + bA|A|^2 + h.o.t.$, coef α and $b \in \mathbb{R}$.
$\alpha > 0$, $b < 0$ when $\Omega_r > 0$, and b changes sign for a small value $\Omega_r < 0$



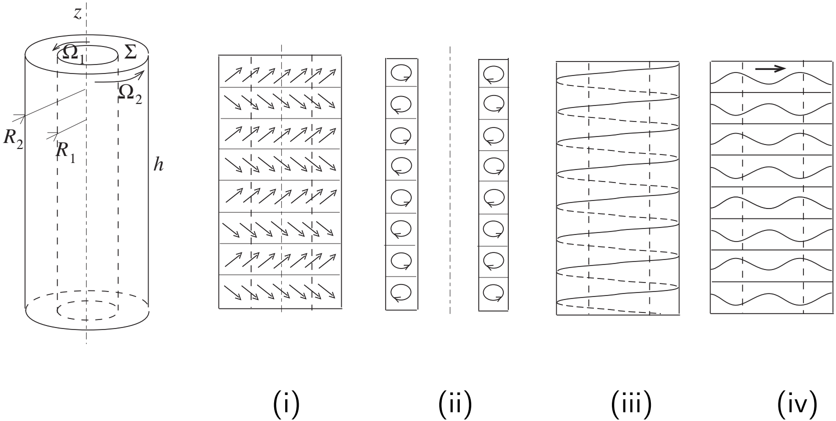
$\mu < 0$



$\mu > 0$ circle of stable equilibria U_ϕ

U_0 and $\boldsymbol{\tau}_{\pi/k_c} U_0 = U_\pi$ invariant under $\mathbf{S}$ implies horizontal cells.

Couette - Taylor problem (6)



(i) Side view of Taylor vortex flow. (ii) Meridian view of Taylor cells.
(iii) Helicoidal waves (traveling in both z and θ directions).
(iv) Ribbons (standing in z direction, traveling in θ direction)

Couette - Taylor problem (7)

Case $\Omega_r < 0$, not too close to 0

Numerical results show that the Couette flow first becomes unstable at a critical value $\mathcal{R}_c$ of $\mathcal{R}$, when a pair of complex conjugate eigenvalues of $\mathbf{L}_{\mathcal{R}}$ crosses the imaginary axis, from the left to the right, as $\mathcal{R}$ is increased, and the rest of the spectrum stays in the left half-complex plane. These two eigenvalues are both double, as this case is generic for $O(2)$ equivariant systems, with two eigenvectors of the form

$$\zeta_0 = e^{i(k_c z + m\theta)} \widehat{U}(r), \quad \zeta_1 = e^{i(-k_c z + m\theta)} \mathbf{S} \widehat{U}(r),$$

where $m \neq 0$ (non-axisymmetric eigenvectors).

Four-dimensional center manifold, and the reduced vector field commute with the actions of symmetries :

$$\boldsymbol{\tau}_a \zeta_0 = e^{ik_c a} \zeta_0, \quad \boldsymbol{\tau}_a \zeta_1 = e^{-ik_c a} \zeta_1, \quad \mathbf{S}\zeta_0 = \zeta_1, \quad \mathbf{S}\zeta_1 = \zeta_0,$$
$$\mathbf{R}_\phi \zeta_0 = e^{im\phi} \zeta_0, \quad \mathbf{R}_\phi \zeta_1 = e^{im\phi} \zeta_1.$$

We are here in the presence of a Hopf bifurcation with $O(2)$ symmetry, with an additional $SO(2)$ symmetry represented by $\mathbf{R}_\phi$.

Couette - Taylor problem (8)

The dynamics are ruled by a system in $\mathbb{C}^2$ of the form

$$\begin{aligned}\frac{dA}{dt} &= AP(|A|^2, |B|^2, \mu) \\ \frac{dB}{dt} &= BP(|B|^2, |A|^2, \mu),\end{aligned}$$

$\mu = \mathcal{R} - \mathcal{R}_c$, and $P(|A|^2, |B|^2, \mu) = i\omega + a\mu + b|A|^2 + c|B|^2 + h.o.t.$ is a smooth function of its arguments, with no “remainder ρ.”
Solutions corresponding to $A = 0$ or to $B = 0$ travel along and around the z-axis with constant velocities. These are *helicoidal waves,* also called *spirals*, and they are axially periodic just as the Taylor vortex flow.
The bifurcating solutions obtained for $|A| = |B|$ are *standing waves* located in fixed horizontal periodic cells, as they are for the Taylor vortex flow, but with a non-axisymmetric internal structure rotating around the axis with a constant velocity. These solutions are also called *ribbons*.

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