Introduction to Hydrodynamic Instabilities

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References

This presentation is based on the book

*Hydrodynamic Instabilities* (F. Charru 2011, Cambridge Univ. Press)

where the references of all the pictures can be found. Other references:

Textbooks on Fluid Mechanics


Specialized Textbooks

1. Instabilities of fluids at rest
Gravity-driven Rayleigh-Taylor instability (1)

Pending drops under a suspended liquid film

Descending fingers of salt water into fresh water
Gravity-driven Rayleigh-Taylor instability (2)

Analysis with viscosity and bounding walls neglected.

Base state:
- fluids at rest with horizontal interface,
- hydrostatic pressure distribution.

Perturbed flow:
Gravity-driven Rayleigh-Taylor instability (3)

Linearized perturbation equations and perturbations $\propto e^{i(kx-\omega t)}$

$\rightarrow$ Dispersion relation $\omega^2 = \frac{(\rho_1 - \rho_2)gk + k^3\gamma}{\rho_1 + \rho_2}$

$\rightarrow$ Instability (complex $\omega$) when $\rho_1 < \rho_2$, with growth rate:

($l_c$ capillary length, $\tau_{ref}$ capillary time)

$\rightarrow$ Long-wave instability
Instabilities related to Rayleigh-Taylor

Inertial instability of accelerated flows (Taylor 1950)

\[ \omega^2 = c_s^2 k^2 - 4\pi G \rho_0. \]

Gravitational instability in astrophysics (Jeans 1902)
Capillary Rayleigh-Plateau instability

jet of water

drops on a spider web

\[ r_-, p^+ \]

\[ r^+, p^- \]

\[ \omega_i \]

\[ \text{ka} \]
Buoyancy-driven Rayleigh-Bénard instability

Linear stability analysis $\rightarrow$ dispersion curve

Bifurcation parameter: Rayleigh number

$$Ra = \frac{\alpha_p g (T_1 - T_2) d^3}{\nu \kappa}$$
A toy-model: convection in an annulus (1)

(Welander 1967)

**Base state:** fluid at rest with temperature

\[
\bar{T} = T_0 - T_1 \frac{z}{a} = T_0 + T_1 \cos \phi.
\]

**Momentum conservation:**

\[
\frac{\partial U}{\partial t} = -\frac{1}{\rho a} \frac{\partial p}{\partial \phi} + \alpha g(T - \bar{T}) \sin \phi - \gamma U.
\]

**Energy conservation:**

\[
\frac{\partial T}{\partial t} + \frac{U}{a} \frac{\partial T}{\partial \phi} = k(T - \bar{T}).
\]

Temperature sought for as

\[
T(t, \phi) = \bar{T} + T_A(t) \sin \phi - T_B(t) \cos \phi,
\]
A toy-model: convection in an annulus (2)

The change of scales

\[ X \propto U, \quad Y \propto T_A, \quad Z \propto T_B, \quad \tau \propto t, \]

then provides the Lorenz system (1963)

\[
\begin{align*}
\partial_\tau X &= -PX + PY \\
\partial_\tau Y &= -Y - XZ + RX \\
\partial_\tau Z &= -Z + XY
\end{align*}
\]

where \( P = k/\gamma, \quad R = \alpha g T_1/2\gamma k a. \)

Stability analysis of the fixed point \((0, 0, 0)\) (fluid at rest)

→ Supercritical pitchfork bifurcation at \( R_{c1} = 1 \) (convection)

Chaotic behavior beyond \( R_{c2}(P) > R_{c1} \) via a subcritical Hopf bifurcation (Lorenz strange attractor).
Thermocapillary Bénard-Marangoni instability
Saffman-Taylor instability of fronts between viscous fluids
2. Stability of open flows: basic ideas
Forced flow: canonic forcings

Consider the (1D) linearized evolution equation for $u(x, t)$

$$L u(x, t) = S(x, t)$$

$L$: differential linear operator involving $x$- and $t$-derivatives

$S(x, t)$: forcing.

Three types of elementary forcing functions of special importance:

$$S(x, t) = F(x)\delta(t) \quad (\text{initial value problem})$$
$$S(x, t) = \delta(x)\delta(t) \quad (\text{impulse response problem})$$
$$S(x, t) = \delta(x)H(t)e^{-i\omega t} \quad (\text{periodic forcing problem})$$

where $\delta$ and $H$ are the Dirac and Heaviside functions.
Impulse response – Definitions

Spatiotemporal evolution of a disturbance localized at $x = 0$ at $t = 0$

(a): Linearly stable flow
(b): Linearly unstable flow, convective instability
(c): Linearly unstable flow, absolute instability
Illustration: waves on a falling film (1)
Illustration: waves on a falling film (2)

A perturbation generated at $x = t = 0$ amplifies while it is convected downstream:

$x = 44$ cm

$x = 97$ cm
Stability criteria

It can be shown that:

- A necessary and sufficient condition for stability is that the growth rates of all the modes with real wavenumber $k$ are negative (temporal stability).
- The criterion for absolute instability is that there exists some wavenumber $k_0$ with zero group velocity and positive growth rate.

A convective instability amplifies any unstable perturbation, and advects it downstream (“noise amplifier”)

An absolutely unstable flow responds selectively to the perturbation with zero group velocity: it behaves like an oscillator with its own natural frequency.
3. Inviscid instability of parallel flows

- Large Reynolds number flows (negligible viscous effects)
- Far from solid boundaries
Illustration 1: tilted channel
(Reynolds 1883, Thorpe 1969)
Illustration 2: wind in a stratified atmosphere
Illustration 3: rising mixing layer
Illustration 4: jet

Jet of carbon dioxide 6 mm in diameter issuing into air at a speed of 40 m s$^{-1}$ (Re = 30 000).
Illustration 5: wake

Wake of a cylinder in water flowing at $1.4 \text{ cm s}^{-1}$ ($\text{Re} = 140$).
General results – Base flow

Ignoring viscous effects, and with unit scales $L$, $V$ and $\rho V^2$, the governing equations are the Euler equations

$$\text{div} \, U = 0,$$
$$\partial_t U + (U \cdot \text{grad}) U = -\text{grad} \, P.$$

These equations have the family of base solutions

$$\overline{U}(x, t) = \overline{U}(y)e_x, \quad \overline{P}(x, t) = \overline{P},$$

corresponding to parallel flow.
General results – Linearized stability problem

Linearized equations for the perturbed base flow \( \overline{U} + u, \overline{P} + p \)

\[
\text{div}\, u = 0, \\
(\partial_t + \overline{U}\partial_x)u + v\partial_y\overline{U}\, e_x = -\text{grad}\, p.
\]

Thanks to the translational invariance in \( t, x \) and \( z \), the solution can be sought in the form of normal modes such as

\[
u(x, t) = \hat{u}(y)e^{i(k_x x + k_z z - \omega t)} + \text{c.c.},
\]

whose amplitudes \( \hat{u}(y), ... \) satisfy the homogeneous system:

\[
\begin{align*}
    ik_x \hat{u} + \partial_y \hat{v} + ik_z \hat{w} &= 0, \\
    i(k_x \overline{U} - \omega)\hat{u} + \partial_y \overline{U}\hat{v} &= -ik_x \hat{p}, \\
    i(k_x \overline{U} - \omega)\hat{v} &= -\partial_y \hat{p}, \\
    i(k_x \overline{U} - \omega)\hat{w} &= -ik_z \hat{p}.
\end{align*}
\]

with the conditions that the perturbations fall off for \( y \rightarrow \pm\infty \) or that \( \hat{v}(y_1) = \hat{v}(y_2) = 0 \) at impermeable walls.
General results – Dispersion relation

The above system can formally be written as the generalized eigenvalue problem

\[ L\phi = \omega M\phi, \]

where \( \phi = (\hat{u}, \hat{v}, \hat{w}, \hat{p}) \) and \( L, M \) linear differential operators.

This problem has a nonzero solution \( \phi \) only if the operator \( L - \omega M \) is noninvertible, \( i.e., \) if for a given wave number the frequency \( \omega \) is an eigenvalue. This condition can be written formally as

\[ D(k, \omega) = 0, \]

which is the dispersion relation of perturbations of infinitesimal amplitude.
General results – Reduction to a 2D problem

Using the Squire transformation

\[ \tilde{k}^2 = k_x^2 + k_z^2, \quad \tilde{\omega} = (\tilde{k}/k_x) \omega, \]
\[ \tilde{k} \tilde{u} = k_x \hat{u} + k_z \hat{w}, \quad \tilde{v} = \hat{v}, \quad \tilde{p} = (\tilde{k}/k_x) \hat{p} \]

the governing equations become, with \( \tilde{c} = c = \omega/k_x \)

\[ i\tilde{k} \tilde{u} + \partial_y \tilde{v} = 0, \]
\[ i\tilde{k}(U - \tilde{c}) \tilde{u} + \partial_y U \tilde{v} = -i\tilde{k} \tilde{p}, \]
\[ i\tilde{k}(U - \tilde{c}) \tilde{v} = -\partial_y \tilde{p}, \]
General results – Squire theorem

Knowing the dispersion relation of the two-dimensional system

\[ \tilde{D}(\tilde{k}, \tilde{\omega}) = 0, \]

the dispersion relation for three-dimensional perturbations can be obtained by means of the Squire transformation:

\[ D(k, \omega) = \tilde{D} \left( \sqrt{k_x^2 + k_z^2}, \frac{\sqrt{k_x^2 + k_z^2}}{k_x} \omega \right) = 0. \]

→ **Squire’s theorem.** For any three-dimensional unstable mode \((k, \omega)\) of temporal growth rate \(\omega_i\) there is an associated two-dimensional mode \((\tilde{k}, \tilde{\omega})\) of temporal growth rate \(\tilde{\omega}_i = \omega_i \sqrt{k_x^2 + k_z^2}/k_x\), which is more unstable since \(\tilde{\omega}_i > \omega_i\).

Therefore when the problem is to determine an instability condition it is sufficient to consider only two-dimensional perturbations.
The Rayleigh equation and inflection point theorem

Introducing the stream function, the 2D stability problem reduces to the Rayleigh equation

\[(\bar{U} - c)(\partial_{yy}\hat{\psi} - k^2\hat{\psi}) - \partial_{yy}\bar{U}\hat{\psi} = 0\]

Thus, if \(\hat{\psi}\) is eigenfunction with eigenvalue \(c\), then so are \(\hat{\psi}^*\) and \(c^*\): stability implies real \(c\) \((c_i = 0\), i.e. neutral stability).

**The Rayleigh theorem.** An inflection point in the velocity profile \(\bar{U}(y)\) is a necessary (but not sufficient) condition for instability.

Assume that the flow is unstable \((c_i \neq 0)\). Divide the Rayleigh equation by \((\bar{U} - c)\), multiply by \(\hat{\psi}^*\), integrate by parts between the walls, with \(\hat{\psi}(y_1) = \hat{\psi}(y_2) = 0\), the imaginary part of the result is

\[c_i \int_{y_1}^{y_2} \frac{\partial_{yy}\bar{U}}{|\bar{U} - c|^2} |\hat{\psi}|^2 dy = 0.\]

Since \(c_i \neq 0\) by assumption, \(\partial_{yy}\bar{U}\) must change sign.
Jump conditions for piecewise-linear velocity profile

The eigenfunctions of the perturbations are exponentials within each layer, and only need to be matched at the discontinuities.

Let $y = y_0 + \eta(x, t)$ be the perturbed position of a discontinuity at $y = y_0$, and $\mathbf{n}$ be the normal. The normal velocity of the fluid must be continuous and equal to the velocity $\mathbf{w} \cdot \mathbf{n}$ of the interface:

$$(\mathbf{U}_+ \cdot \mathbf{n})(y_0 + \eta) = (\mathbf{U}_- \cdot \mathbf{n})(y_0 + \eta) = \mathbf{w} \cdot \mathbf{n}.$$  

Linearizing at $y = y_0$, with $\mathbf{n} = (-\partial_x \eta, 1)$ and $\mathbf{w} \cdot \mathbf{n} = -\partial_t \eta$, introducing the normal modes and eliminating $\hat{\eta}$ gives:

$$\Delta \left( \frac{\hat{\psi}}{\mathbf{U} - c} \right) = 0, \quad \text{where} \quad \Delta[X] = X_+(y_0) - X_-(y_0).$$

The continuity of pressure gives similarly

$$\Delta[(\mathbf{U} - c)\partial_y \hat{\psi} - \partial_y \mathbf{U} \hat{\psi}] = 0.$$  

→ Complete determination of the eigenfunctions.
Kelvin-Helmholtz instability of a vortex sheet

The solution of the Rayleigh equation, \( \hat{\psi}_j = A_j e^{-ky} + B_j e^{ky}, \ j = 1, 2, \) the fall-off of the perturbations at infinity (\( A_1 = 0 \) and \( B_2 = 0 \) for \( k > 0 \)), and the jump conditions at the interface gives

\[
(U_1 - c)A_2 - (U_2 - c)B_1 = 0 \\
(U_2 - c)A_2 + (U_1 - c)B_1 = 0
\]

which has a nontrivial solution only when (dispersion relation)

\[
(U_1 - c)^2 + (U_2 - c)^2 = 0,
\]

i.e. \( c = \frac{\omega}{k} = U_{av} \pm i\Delta U, \) with \( 2U_{av} = U_1 + U_2, \) \( 2\Delta U = U_1 - U_2. \)
Mechanism of the Kelvin-Helmholtz instability

“Bernoulli effect”
Kelvin-Helmholtz with vorticity layer of finite thickness $2\delta$

→ Stable short waves, and long-wave instability with $\omega_{i,\text{max}} \approx 0.2U/\delta$

Inviscid analysis valid whenever $\delta/\Delta U \ll \delta^2/\nu$, i.e. $Re \gg 1$.
Viscous effects decrease $\omega_i$ and $k_{\text{cutoff}}$ (also increase $\delta$).
Couette-Taylor centrifugal instability

The Couette flow between two coaxial cylinders may be unstable due to the centrifugal force.

Rayleigh (1916): a stable stratification of centrifugal force satisfies $\Omega_1 a_1^2 < \Omega_2 a_2^2$.

Landmark experiments by G. I. Taylor (1923)
1: inner
2: outer
$a_2/a_1$ fixed
4. Viscous instability of parallel flows

- Boundary layers and Poiseuille flow have no inflection point.
- However, experiments show that they may be unstable...
Illustration 1: Poiseuille flow in a tube
(Reynolds 1883)
Illustration 2: Boundary layer

Tollmien-Schlichting waves
General results – 2nd Squire’s theorem

- The analysis goes along the same lines as for inviscid flow, with the viscous diffusion term
  \[ \frac{1}{Re} \Delta \mathbf{U}, \quad Re = \frac{UL}{\nu} \]

- The generalized eigenvalue problem has nontrivial solution when the operator is singular, i.e. \( D(k, \omega, Re) = 0 \).

- Squire’s theorem. For any unstable oblique mode \((k, \omega)\) of temporal growth rate \(\omega_i\) for Reynolds number \(Re\) it is possible to associate a two-dimensional mode \((\tilde{k}, \tilde{\omega})\) of temporal growth rate \(\tilde{\omega}_i = \omega_i \sqrt{k_x^2 + k_z^2}/k_x\), higher than \(\omega_i\), at a Reynolds number \(\tilde{Re} = Re k_x/\sqrt{k_x^2 + k_z^2}\), lower than \(Re\).

Corollary. If there exists some \(Re_c\) above which a flow is unstable, the destabilizing normal mode for \(Re = Re_c\) is 2D.
The Orr-Sommerfeld equation
(Orr 1907, Sommerfeld 1908)

For plane flow, 2D disturbances obey the Orr-Sommerfeld equation (Rayleigh equation + viscous term):

\[(\bar{U} - c)(\partial_{yy} - k^2)\hat{\psi} - \partial_{yy}\bar{U}\hat{\psi} = \frac{1}{i k \text{Re}} (\partial_{yy} - k^2)^2 \hat{\psi}.\]

- No exact solution except for linear velocity profile (integrals of Airy functions)
- Difficult to solve for high \(\text{Re}\), especially near the critical layer (where \(c = \bar{U}\)) and near walls or interfaces
- May be solved analytically using perturbation methods (small or large \(k\), small or large \(\text{Re}\))
- May be solved numerically using shooting or spectral methods

Eigenfunctions:
Spatial evolution of the forced disturbances ... for increasing $\text{Re} = 4000 \ldots 8000$ ... for varying frequency

The spatial and temporal growth rates are related through the group velocity:

$$\omega_i^T = -c_g k_i^S$$

(Gaster 1962)
Plane Poiseuille flow (3)

Spatial growth rates versus frequency $\beta = 2\pi fU/h$
for increasing $Re = 3000...7000$ and comparison with calculations
Plane Poiseuille flow (4)

Stability diagram

$\text{Stable}$

$\text{Unstable}$

$k_r h$

$\text{Re} \times 10^{-3}$
Transient growth

When the disturbance is not well controlled (amplitude $\gtrsim 1\%$), instability occurs well below $\text{Re} = 5772$.

An explanation relies on the non-normality of the Orr-Sommerfeld operator and the associated Orr equation for the $y$-vorticity, which implies transient growth of the superposition of stable eigenmodes, and may trigger nonlinear behaviour.

Longitudinal vortices give rise to the strongest transient growth (optimal perturbation).
Plane Poiseuille flow: tentative bifurcation diagram

\[ kh = 1.32 \]

\[ kh = 1.02 \]
Poiseuille flow in a pipe

... is linearly stable!

Difficulties begin...
Boundary layer on a flat plate (1)

Velocity fluctuations of a forced Tollmien–Schlichting wave, measured at different positions (in feet) downstream from the leading edge, for upstream velocity \( U_\infty = 36.6 \text{ m s}^{-1} \) (Schubauer & Skramstad 1947).
Although the flow is not strictly parallel ($\delta(x)$ increases), local analysis is possible with $U(x, y)$, and $x$ treated as a parameter.
Boundary layer on a flat plate (3)

Marginal stability: (- -) nonparallel theory, (×) measurements, (●) DNS.
5. Nonlinear dynamics with few degrees of freedom

- What happens beyond the exponential growth, when nonlinear terms are no longer negligible?
- No general theory
- ‘Weakly nonlinear analysis’ particularly important owing to its fairly general nature based on perturbation methods.
Beyond the exponential growth: the Landau equation

According to the linear stability theory, a perturbation of a base flow can be expressed as a sum of uncoupled eigenmodes:

$$u(x, t) = \frac{1}{2} \left( A(t)f(x) + A^*(t)f^*(x) \right)$$

$$f(x) : \text{spatial structure of the mode, } A(t) \text{ its time evolution.}$$

Basic idea (Landau 1944): $A(t)$ grows exponentially as the solution of

$$\frac{dA}{dt} = \sigma A, \quad \sigma \quad \text{temporal growth rate}$$

This equation can be viewed as a Taylor series expansion of $dA/dt$ in powers of $A$, truncated at first order. For problems invariant under time translation, the equation for $A$ must be invariant under the rotations $A \to Ae^{i\phi}$. The lowest order term satisfying this condition is $|A|^2 A$. Hence the Landau equation

$$\frac{dA}{dt} = \sigma A - \kappa |A|^2 A.$$
Calculations of the Landau constants

The Landau constant has been calculated for the major instabilities, through the expansion of the governing equations in power series of the amplitude.

- Rayleigh-Bénard problem: $\kappa_r$ is positive, corresponding to supercritical pitchfork bifurcation (Gor’kov 1957, Malkus & Veronis 1958)
- Taylor-Couette flow: same conclusion (see Chossat & Iooss 1994)
- Plane Poiseuille flow: the instability at $\text{Re} = 5772$ is subcritical: no saturation by the cubic term (Stuart 1958, Watson 1960). More work is needed...

The expansion procedure is illustrated below with nonlinear oscillators.
Van der Pol oscillator: saturation of the amplitude (1927)

\[
\frac{d^2u}{dt^2} - (2\epsilon\mu - u^2) \frac{du}{dt} + \omega_0^2 u = 0, \quad \mu = \mathcal{O}(1), \quad \epsilon \ll 1.
\]

The fixed point \((u, du/dt) = (0, 0)\) is a stable focus for \(\mu < 0\), unstable for \(\mu > 0\).

- Growth rate \(\epsilon\mu \ll \omega_0\): slow variation of the amplitude expected
- For \(\mu > 0\), saturation expected for \(u \sim \epsilon^{1/2}\).

Hence, \(u(t)\) sought for as (multiple scale expansion)

\[
u(t) = \epsilon^{1/2} \tilde{u}(t, T), \quad T = \epsilon t, \quad \tilde{u} = u_0 + \epsilon u_1 + \epsilon^2 u_2 + ...
\]
Van der Pol: solution at order $\epsilon^0$

At the dominant order, the linear problem to solve is

\[ Lu_0 = 0 \quad \text{with} \quad L = \frac{\partial^2}{\partial \tau^2} + \omega_0^2 \]

with solution

\[ u_0 = \frac{1}{2} \left( A(T)e^{i\omega_0 \tau} + A(T)^* e^{-i\omega_0 \tau} \right). \]
Van der Pol: solution at order $\epsilon^1$

At the next order, the linear nonhomogeneous problem to solve is

$$Lu_1 = -2 \frac{\partial^2 u_0}{\partial \tau \partial T} + (2\mu - u_0^2) \frac{\partial u_0}{\partial \tau}$$

with r.h.s. known from the previous step, so that:

$$Lu_1 = i\omega_0 (\mu A - \frac{dA}{dT}) e^{i\omega_0 \tau} - \frac{i\omega_0}{8} (|A|^2 A e^{i\omega_0 \tau} + A^3 e^{3i\omega_0 \tau}) + \text{c.c.},$$

Cancellation of the resonant forcing (solvability condition) leads to

$$\frac{dA}{dT} = \mu A - \kappa |A|^2 A, \quad \kappa = \frac{1}{8} \quad \text{(Landau equation)}$$

$$A = a(T)e^{i\phi(T)} \quad \rightarrow \quad \frac{da}{dT} = \mu a - \kappa a^3, \quad \frac{d\phi}{dT} = 0.$$

- Supercritical Hopf bifurcation at $\mu = 0$
- The nonlinearity saturates the amplitude.
Van der Pol: asymptotic vs. numerical solutions

(a) Landau equation

(b) Numerical solution

Farther from threshold
Duffing oscillator: frequency correction

\[ \frac{d^2 u}{dt^2} + u + \epsilon u^3 = 0, \quad \mu = O(1), \quad \epsilon \ll 1. \]

Can be written:

\[ \frac{d^2 u}{dt^2} = -V'(u), \quad V(u) = \frac{u^2}{2} + \epsilon \frac{u^4}{4}, \]

\[ \varepsilon = 0.1 \]

\[ \varepsilon = -0.1 \]
Duffing: multiple scale analysis

Expand as before

\[ u(t) = u_0(\tau, T) + \epsilon u_1(\tau, T) + ... \]

\[ \rightarrow Lu_0 = 0 \quad \text{with} \quad L = \frac{\partial^2}{\partial \tau^2} + 1, \]

\[ Lu_1 = -2 \frac{\partial^2 u_0}{\partial \tau \partial T} - u_0^3. \]

The solvability condition at order \( \epsilon \) gives the Landau equation

\[ \frac{dA}{dT} = \frac{3i}{8} |A|^2 A \quad \text{(no linear term)} \]

\[ A = a(T)e^{i\phi(T)} \quad \rightarrow \quad \frac{da}{dT} = 0, \quad \frac{d\phi}{dT} = \frac{3}{8} a^2. \]

Hence the final solution:

\[ u(t) = a_0 \cos(\omega t + \phi_0) + \mathcal{O}(\epsilon), \quad \omega = 1 + \frac{3}{8} \epsilon a_0^2 + \mathcal{O}(\epsilon^2). \]
Duffing: asymptotic vs. numerical solutions

dashed: $O(\epsilon^0)$ solution

dashed-dotted: $O(\epsilon^1)$ solution

plain curve: numerical solution
Derivation of the Landau equation from the Kuramoto-Sivashinsky (KS) equation

\[ \partial_t u + 2V u \partial_x u + R \partial_{xx} u + S \partial_{xxxx} u = 0. \]

Normal modes \( \propto e^{\sigma t + i(\omega t - kx)} \) → dispersion relation:

\[ \sigma = R k^2 - S k^4, \quad \omega = 0 \]
KS: amplitude expansion

Search for periodic solutions with wavelength $L = 2\pi/k_1$. Rescale $u, x$ and $t$ so that $k_1 = 1, S = 1$, and expand in Fourier series

$$u(x, t) = \frac{1}{2} \sum_{n=-N}^{N} A_n(t) e^{inx}, \quad \text{with} \quad A_{-n} = A_n^*.$$

Assume $A_n \sim \epsilon^n$ (to be checked a posteriori), and keep the first three harmonics:

$$\frac{dA_1}{dt} = \sigma_1 A_1 - iVA_1^* A_2 + O(A_1^5),$$
$$\frac{dA_2}{dt} = \sigma_2 A_2 - iVA_1^2 + O(A_1^4),$$
$$\frac{dA_3}{dt} = \sigma_3 A_3 - 3iVA_1 A_2 + O(A_1^5).$$
KS: Reduction to the central manifold

Close to threshold, \( \frac{d}{dt} \sim \sigma_1 \ll 1 \), and \( |\sigma_n| \gg \sigma_1 \), so that

\[
A_2 = \frac{iV}{\sigma_2} A_1^2 + \mathcal{O}(A_1^4),
\]

\[
A_3 = \frac{3iV}{\sigma_3} A_1 A_2 + \mathcal{O}(A_1^5) \sim -\frac{3V^2}{\sigma_2 \sigma_3} A_1^3.
\]

\( \rightarrow \) All the harmonics are ‘slaved’ to the fundamental.

The dynamics of the fundamental is governed by the Landau equation

\[
\frac{dA_1}{dt} = \sigma_1 A_1 - \kappa |A_1|^2 A_1 + \mathcal{O}(A_1^5), \quad \kappa = -\frac{V^2}{\sigma_2} > 0
\]

\( \rightarrow \) Supercritical Hopf bifurcation at \( R = 1 \).
Illustration: waves at a sheared interface (Barthelet, Charru & Fabre 1995)

Two-layer Couette flow experiments in a annular channel, of mean radius $R = 0.4$ m. The interface between the two viscous fluids becomes unstable beyond some critical upper plate velocity $U$: a long wave grows with $\lambda = 2\pi R$.

Saturated wave just below the threshold, just above, and farther:
Sheared interface (2): bifurcation diagram

Bifurcation diagram (no hysteresis), and saturation time:

![Bifurcation diagram](image1)

![Saturation time](image2)
Sheared interface (3): dynamics of the harmonics

Check that $A_2 \propto A_1^2$ and $A_3 \propto A_1^3$ as predicted by the theory?

Time evolution of the harmonics $\frac{1}{2} A_n(t)e^{in(k_1x-\omega_1^0 t)} + \text{c.c.}$, with amplitudes $A_n(t) = |A_n(t)|e^{i\phi_n(t)}$, obtained by pass-band filtering about the frequency $n\omega_1^0$. Modulus $|A_n(t)|$ and slow phases $\phi_n(t)$ obtained by Hilbert transform:

- $|A_2|/|A_1|^2$
- $|A_3|/|A_1|^3$
- $\phi_2 - 2\phi_1$
- $\phi_3 - 3\phi_1$
Sheared interface (4): experimental center manifold

\[ \frac{A_2}{A_{sat}} \]

\[ |A_1|/|A_{1, sat}| \]
Sheared interface (5): farther from threshold...

\[ \frac{A}{A_{sat}} \]

Saturated wave:
\[ \lambda = 2\pi R \text{ or } \frac{1}{2} 2\pi R \]
6. Nonlinear dispersive waves

- Surface gravity waves of amplitude 'not small' are not sinusoidal
- The dispersion relation $\omega_0^2 = gk_0$ is not accurately satisfied
- How harmonics can propagate with the same velocity as the fundamental?
- What is the stability of finite amplitude waves?
Finite amplitude gravity waves: Stokes 1847

Using a series expansion in powers of the wave slope $\epsilon = k_0 a_0$, Stokes (1847) found the profile of the free surface $\eta(x, t)$

$$\frac{\eta(x, t)}{a_0} = \frac{\epsilon}{2} + \cos \theta + \frac{\epsilon}{2} \cos 2\theta + \frac{3\epsilon^2}{8} \cos 3\theta + O(\epsilon^3),$$

with phase $\theta = k_0 x - \omega t$ and frequency

$$\omega = \omega_0 \left(1 + \frac{1}{2} \epsilon^2 + O(\epsilon^4)\right), \quad \omega_0^2 = g k_0.$$
The Stokes wave is unstable
(Benjamin & Hasselman 1967)

The progressive wave train ($\lambda = 2.2$ m), degenerates into a series of wave groups, and eventually disintegrates:

near the wave-maker

60 m downstream
The Benjamin-Feir instability
(Benjamin & Feir 1967)

The instability of gravity waves is a generic instability of dispersive waves, of wave number $k_0$, to perturbations with nearby wave numbers $k_0 + \delta k$, now known as a side-band instability.

These perturbations grow exponentially via a resonance mechanism when

$$\frac{\delta k^2}{k_0^2} < 8(k_0 a_0)^2,$$

The two most highly amplified perturbations are those with wave numbers $k_0(1 \pm 2k_0 a_0)$, and their growth rate is

$$\sigma_{\text{max}} = \frac{\omega_0}{2} (k_0 a_0)^2.$$
Experimental validation of the theory
(Lake & Yuen 1977)
Experimental validation of the theory (2)
(Lake & Yuen 1977)
Model problem: a chain of coupled oscillators

In the long-wave limit and with appropriate choice of the time, mass and length scales, the equation of motion reduces to the nonlinear Klein-Gordon equation

\[
\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = -V'(u), \quad V(u) = \frac{u^2}{2} + \gamma u^4, \quad \gamma = \frac{1}{24}
\]

Dispersion relation of waves with infinitesimal amplitude (no instability):

\[
\omega^2 = 1 + k^2.
\]
Model problem (1): nonlinear Klein-Gordon wave

Seek a traveling wave solution propagating in the \( x \)-direction (\( c = \omega / k > 0 \)) as

\[
u(x, t) = \frac{1}{2} \sum_{n=-N}^{N} \epsilon A_n(t) e^{i(k_n x - \omega_n t)},
\]

The time scale of the nonlinear interactions is of order \( \epsilon^{-2} \). Introducing the slow time scale \( T = \epsilon^2 t \), we obtain the amplitude equation for the \( n \)th mode:

\[
\frac{dA_n}{dT} = -\frac{i \gamma}{2\omega_n} \sum_{k_p + k_q + k_r = k_n} A_p A_q A_r e^{i(\omega_n - \omega_p - \omega_q - \omega_r) T / \epsilon^2}.
\]

This interaction leads to remarkable solutions, in particular, when the frequencies satisfy the very special \textit{resonance condition}

\[
\omega_p + \omega_q + \omega_r = \omega_n.
\]
Model problem (2): nonlinear Klein-Gordon wave

Let us consider the resonant interaction of a wave of wave number $k_0$ with itself (self-interaction). The summation runs over $2^3$ triads $(\pm k_0, \pm k_0, \pm k_0)$, only three satisfy the resonance condition: $(k_0, k_0, -k_0)$, $(k_0, -k_0, k_0)$, and $(-k_0, k_0, k_0)$.

The amplitude equation for $A_0$ then reduces to

$$\frac{dA_0}{dT} = -i\beta A_0^2 A_0^*, \quad \beta = \frac{3\gamma}{2\omega_0}, \quad \omega_0 = \sqrt{1 + k_0^2}.$$ 

with solution $A_0 = a_0 e^{-i\beta a_0^2 T}$, $a_0 = O(1)$ real.

Returning to the original angular variable

$$u(x, t) = \epsilon a_0 \cos(k_0 x - \omega t) + O(\epsilon^3), \quad \omega = \omega_0 + \beta(\epsilon a_0)^2$$

The frequency and speed of the wave are modified by the self-interaction due to the cubic nonlinearity: they depend on the amplitude. The frequency correction is the same as that of a Duffing oscillator.
Model problem (3): stability of the nonlinear wave

Consider the effect of a perturbation of the monochromatic wave in the form of two waves with wave numbers close to $k_0$:

$$k_0 \pm \epsilon K \text{ with } K = \mathcal{O}(1), \quad \text{frequencies } \omega_{\pm}, \quad \text{amplitudes } |A_{\pm}| \ll |A_0|.$$  

Keeping only the dominant terms, the amplitude equations reduce to

$$\frac{dA_-}{dT} = -i \beta a_0^2 \left(2A_- + A_+^* e^{i(\Omega - 2\beta a_0^2)T} \right)$$
$$\frac{dA_+}{dT} = -i \beta a_0^2 \left(2A_+ + A_-^* e^{i(\Omega - 2\beta a_0^2)T} \right).$$

with $\Omega = \frac{1}{\epsilon^2} (\omega_+ + \omega_- - 2\omega_0) \approx \omega'' K^2 = \mathcal{O}(1), \quad \omega'' = \frac{\partial^2 \omega}{\partial k^2} (k_0) = \omega_0^{-3}$.  

This system can be made autonomous by a rotation of $A_{\pm}$ in the complex plane, and has nontrivial solutions $\propto e^{\sigma T}$ if (dispersion relation)

$$\sigma^2 + \beta \omega'' a_0^2 K^2 + \frac{\omega''^2}{4} K^4 = 0.$$
A necessary condition for instability ($\sigma_r \neq 0$) is $\beta \omega_0'' < 0$.

Then, the wave is unstable to side-band perturbations of wave numbers

$$k = k_0 \pm \epsilon K \text{ with } K < K_{\text{off}} = 2a_0 \sqrt{-\beta/\omega_0''}.$$  

The two most amplified wave numbers are

$$k_{\text{max}} = k_0 \pm \frac{1}{\sqrt{2}} \epsilon K_{\text{off}}.$$
The Stokes wave instability revisited

The instability condition $\beta \omega_0'' < 0$ established for the Klein–Gordon wave is actually general: it is valid for any dispersive nonlinear wave, with $\beta$ the coefficient of the nonlinear correction $\beta(\epsilon a_0)^2$ to the frequency.

For example, for a gravity wave we obtain from the dispersion relation for finite-amplitude waves found by Stokes:

$$\omega_0'' = -\frac{\omega_0}{4k_0^2}, \quad \beta = \frac{1}{2} \omega_0 k_0^2.$$  

The instability condition is then

$$(\epsilon K)^2 < 8k_0^4(\epsilon a_0)^2$$

identical to that obtained by Benjamin and Feir (1967) in their solution of the hydrodynamical problem! This is no accident, as the instability results from a competition between the linear dispersion and the nonlinearity, the effect of the latter being contained entirely in the nonlinear correction of the wave frequency.
Alternative analysis: dynamics of a wave packet

(Benney & Newell 1967; Stuart & DiPrima 1978)

A wave packet centered on the wave number $k_0$ propagating in the direction of increasing $x$ can be represented as the Fourier integral

$$u(x, t) = \frac{1}{2} A(x, t) e^{i(k_0 x - \omega_0 t)} + c.c.$$  

where $\omega_0 = \omega(k_0)$ (real) and the envelope $A(x, t)$ of the wave packet is defined as

$$A(x, t) = \int_0^{+\infty} \hat{u}(k) e^{i(k-k_0)x - i(\omega(k)-\omega_0)t} \, dk.$$  

Expand $\omega(k)$ in Taylor series about $k_0$ and truncate at second order:

$$\omega - \omega_0 = c_g(k-k_0) + \frac{\omega''_0}{2}(k-k_0)^2 \quad \text{with} \quad c_g = \frac{\partial \omega}{\partial k}(k_0), \quad \omega''_0 = \frac{\partial^2 \omega}{\partial k^2}(k_0).$$

We recognize the general solution of the envelope equation

$$i \frac{\partial A}{\partial t} = -ic_g \frac{\partial A}{\partial x} + \alpha \frac{\partial^2 A}{\partial x^2}, \quad \alpha = \frac{1}{2} \omega''_0.$$
Nonlinear dynamics: the nonlinear Schrödinger equation

According to the above linear envelope equation, the width of the wave packet increases linearly with time due to dispersion, while its amplitude decreases as $1/\sqrt{t}$. Nonlinearity may counteract dispersion.

For problems invariant under translations $x \rightarrow x + \xi$ and $t \rightarrow t + \tau$, the nonlinear envelope equation must be invariant under the transformation $A \rightarrow Ae^{i\phi}$. Hence the nonlinear Schrödinger (NLS) equation:

$$i \frac{\partial A}{\partial t} = -ic_g \frac{\partial A}{\partial x} + \alpha \frac{\partial^2 A}{\partial x^2} - \beta |A|^2 A.$$  

If the problem is invariant under reflections $x \rightarrow -x$ and $t \rightarrow -t$, $\beta$ is real.

For the coupled pendulum problem, a multiple scale analysis shows $\beta = 3\gamma/2\omega_0$. 

Stability of a quasi-monochromatic wave (1)

The nonlinear Schrödinger equation admits the spatially uniform solution

\[ A_0 = a_0 e^{i(\Omega t + \Phi)}, \quad a_0 = |A_0| \text{ real}, \quad \Omega = \beta a_0^2, \]

which corresponds to the unmodulated traveling wave

\[ u(x, t) = a_0 \cos(k_0 x - \omega t + \Phi), \quad \omega = \omega_0 + \beta a_0^2, \]
Stability of a quasi-monochromatic wave (2)

Perturb $A_0$ as

$$A(x, t) = (a_0 + a(x, t))e^{i(\Omega t + \Phi + \varphi(x, t))}$$

substitute in the NLS, linearize and separate the real and imaginary parts:

$$\partial_t a = \alpha a_0 \partial_{xx} \varphi,$$
$$\partial_t \varphi = 2 \beta a_0 a - (\alpha / a_0) \partial_{xx} a.$$

This linear system admits solutions of the form $e^{\sigma t - ipx}$, with (dispersion relation):

$$\sigma^2 + 2 \alpha \beta a_0^2 p^2 + \alpha^2 p^4 = 0.$$

The Benjamin-Feir (side-band) instability is exactly recovered.
7. Nonlinear dynamics of dissipative systems

- What happens when the size of the system is large compared with the wavelength of an unstable mode for $R \approx R_c$?
- We first consider systems with the translational and reflectional symmetries.
- We then consider propagating waves (no reflectional symmetry).
Nonlinear dynamics of dissipative systems

Linear analysis, for $\omega = \partial \omega / \partial k = 0$ at $(k, R) = (k_c, R_c)$

Close to threshold ($\epsilon^2 = r - r_c \ll 1$ with $r = R/R_c$), expand the growth rate:

$$\tau_c \sigma(k, r) = (r - r_c) - \xi_c^2 (k - k_c)^2 + ... ,$$

where $\tau_c$ and $\xi_c$ are characteristic time and length scales defined as

$$\frac{1}{\tau_c} = \frac{\partial \sigma}{\partial r}, \quad \frac{\xi_c^2}{\tau_c} = -\frac{1}{2} \frac{\partial^2 \sigma}{\partial k^2}.$$
Dynamics of a wave packet

The perturbation of the base state can be written as

\[ u(x, t) = \frac{1}{2} A(x, t) e^{i k_c x} + \text{c.c.}, \]

where the envelope \( A(x, t) \) of the wave packet is defined as

\[ A(x, t) = \int_0^{+\infty} \hat{u}(k) e^{i(k - k_c)x + \sigma(k)t} \, dk. \]

Replacing \( \sigma(k) \) by its Taylor series, we recognize the general solution of the envelope equation

\[ \tau_c \frac{\partial A}{\partial t} = (r - r_c) A + \xi^2 \frac{\partial^2 A}{\partial x^2}. \]

For systems invariant under space and time translation, the weakly nonlinear dynamics is governed by the Ginzburg–Landau envelope equation with real \( \kappa \):

\[ \tau_c \frac{\partial A}{\partial t} = (r - r_c) A + \xi^2 \frac{\partial^2 A}{\partial x^2} - \kappa |A|^2 A, \]
Saturated pattern, and linear stability (1)

**Periodic pattern.** For $\kappa > 0$, the Ginzburg–Landau equation possesses a continuous family of uniform, stationary solutions

$$U_0(x, t) = u_0 \cos(k_0 x + \Phi)$$

of amplitude $u_0$ and wave number $k_0$ defined as

$$u_0 = \sqrt{r - r_c} \sqrt{\frac{1 - q_0^2}{\kappa}}, \quad k_0 = k_c + \frac{\epsilon q_0}{\xi_c}, \quad -1 \leq q_0 \leq 1.$$  

**Stability.** Perturb the amplitude as $a_0 + \tilde{a}(X, T)$ and the phase as $\Phi + \varphi(X, T)$, linearize, and find

$$\partial_T \tilde{a} = -2a_0^2 \tilde{a} + \partial_{xx} \tilde{a} - 2a_0 q_0 \partial_x \varphi,$$

$$\partial_T \varphi = -\frac{2q_0}{a_0} \partial_x \tilde{a} + \partial_{xx} \varphi.$$  

This system admits solutions $\propto e^{ipX + \sigma T}$, with (dispersion relation)

$$\sigma_{\pm} = -(a_0^2 + p^2) \pm \sqrt{a_0^4 + 4q_0^2 p^2}.$$
The amplitude mode is stable ($\sigma_- < 0$), and slaved to the phase mode which is unstable ($\sigma_+ > 0$) for $q_0^2 > 1/3$.

It can be shown that the instability is subcritical: no saturation mechanism.
The roll pattern, initially ‘compressed’ (thermal impression technique), relaxes to larger wavelength through a ‘cross-roll’ instability.

$t = t_0$

$t = t_0 + 52 \text{ mn}$
Illustration: Rayleigh-Bénard convection (2)

The roll pattern, initially ‘stretched’ (thermal impression technique), relaxes to smaller wavelength through a ‘zig-zag’ instability.

$t = t_0$

$t = t_0 + 127 \text{ mn}$
Travelling dissipative waves

Consider a wave packet near the instability threshold ($\sigma = 0$ and $\omega = \omega_c$ at the critical point ($k_c, R_c$)), expand the dispersion relation, take the inverse Fourier transform, add the dominant nonlinear term $|A|^2 A$.

We obtain the complex Ginzburg-Landau (CGL) equation

$$\tau_c \left( \frac{\partial A}{\partial t} + c_g \frac{\partial A}{\partial x} \right) = (r - r_c)A + (\xi_c^2 + \frac{i \tau_c \omega''}{2}) \frac{\partial^2 A}{\partial x^2} - \kappa |A|^2 A,$$
Finite-amplitude travelling waves, and their stability

The CGL equation possesses a continuous family of travelling wave solutions. Instability corresponds to negative diffusion in the equation of the phase perturbation:

\[
D(q_0) = 1 + c_1 c_2 - 2q_0^2 (1 + c_2^2)/(1 - q_0^2), \quad -1 \leq q_0 \leq 1.
\]

The Benjamin-Feir and Eckhaus instability are unified (Stuart & DiPrima 1978).